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A Simplified Method of Moment (MoM) Approach to solving nth Order Linear Differential Equations

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Abstract—The Method of Moment (MoM) is a very successful tool in solving complex geometry electromagnetic (EM) problems, and is considered as the first computational approach in solving these problems. This paper approaches MoM in a very simple way and aims to produce a valuable procedure to solving boundary value problems, especially nth differential equations. Starting with the definition of the method, the problem is converted into its respective integral equation form, and then the unknown function is expanded into a sum of weighted basis functions, where the weight coefficients are to be found. The Galerkin method, which selects testing functions equal to the basic functions, is adopted. The problem then becomes a system of linear equations, which is solved analytically or numerically to find the needed weight coefficients. Two examples, a second- and a third-order differential equations, are considered to illustrate the application of the procedure, which can be used as well for the solution of other boundary-value problems. The considered examples show the detailed calculation process and make it easier to understand the solution procedure.

Index Terms—Basis functions, Differential equations, Method of Moment, Galerkin method, Weight coefficient

1 Introduction

BEFORE digital computers, the design and analysis of electromagnetic devices and structures were largely experimental. This changed after computers and programming languages appeared, where researchers began using them to challenge electromagnetic problems that could not be solved analytically. This led to a burst of development in a new field called computational electromagnetics (CEM). The Method of Moment (MoM) and other powerful numerical analysis techniques have been developed in this area in the last 50 years [1]. MoM, described in [2], is a simple numerical technique used to convert integro-differential equations into a linear system that can be solved numerically using a computer. When the order of the equation is small, MoM can analytically solve this problem in a general and very clear manner.

A large number of publications, including textbooks, graduate theses, and journal papers have been dedicated to MoM, but in most part these have been part of graduate courses on computational electromagnetics, or aimed to help professionals apply MoM in their field problems. It is unusual to see a work on MoM addressed to undergraduate students to aid their understanding of this method, and help them apply it to solve problems they face in their undergraduate courses. This is reflected in the publications references hereafter. In [3], it is shown that the inner product involved in MoM is usually an integral, which is evaluated numerically by summing the integrand at certain discrete points. Three simple examples on the use of MoM in electromagnetics are presented in [4]. These examples deals with the input impedance of a short dipole, a plane wave scattering from a short dipole, and two coupled short dipoles. In [5], MoM is introduced in a way to give a deeper insight into electromagnetic phenomena. This is done by presenting examples and software programs, and also by giving the curriculum needed to quickly learn the basic concepts of numerical solutions.

This paper approaches MoM in a very simple way and aims to produce a valuable procedure to solving boundary value problems, especially differential equations. Starting with the definition of the method, the problem is converted into its respective integral equation form, and then the unknown function is expanded into a sum of weighted basis functions, where the weight coefficients are to be found. The Galerkin method, which selects testing functions equal to the basis functions, is adopted. The problem then becomes a system of linear equations, which is solved analytically or numerically to find the needed weight coefficients. Two examples, a second- and a third-order differential equations, are considered to illustrate the application of the procedure, which can be used as well for the solution of other boundary-value problems.

2 Method of Moments Solution Approach

For students to be familiarized with MoM, it is good to introduce them to new terms such as expansion and testing. This is because the MoM method starts by expanding the unknown quantity, which is to be solved for, into a set of known functions with unknown coefficients. The resulting equation is then converted into a linear system of equations by enforcing the boundary conditions at a number of points. This resulting linear system is then solved analytically for the unknown coefficients. It is here to note that such an approach is very simple and quite interesting when applied to differential equation of order less than 3, but it is applicable for equations of higher order.

Accordingly, it is advisable to start with some basic mathematical techniques for reducing functional equations to matrix equations. A deterministic problem is considered, which will be solved by reducing it to a suitable matrix equation, and hence the solution could be found by matrix inversion. The examples that we will choose are simple and easily illustrate the theory without any complicated mathematics. Linear spaces and operators will be used in our solution. At first, it is recommended to introduce MoM and define some terms related to first order non-homogeneous differential equation. The choice of this equation is important only for better understanding of the solution.

A general n^{th} order linear differential equation, defined over a domain D, has the form

$$a_n \frac{d^n f(x)}{dx^n} + a_{n-1} \frac{d^{n-1} f(x)}{dx^{n-1}} + \dots + a_1 \frac{df(x)}{dx} + a_0 f(x) = g(x) \quad (1)$$

In (1), the coefficients $a_n, a_{n-1}, \dots, a_1, a_0$ and $g(x)$ are known quantities, and $f(x)$ is the function whose solution is to be determined. Equation (1) can be written in the form of an operator equation

$$L(f(x)) = g(x), \quad (2)$$

where L is the operator equation, operating on $f(x)$, and given by

$$L = a_n \frac{d^n \dots}{dx^n} + a_{n-1} \frac{d^{n-1} \dots}{dx^{n-1}} + \dots + a_1 \frac{d \dots}{dx} + a_0 \dots \quad (3)$$

The solution of (1) is based on defining the inner product $< f, g >$, a scalar quantity valid over the domain of definition of L, which is given by

$$\langle f(x), g(x) \rangle = \int_D f(x) g(x) dx \quad (4)$$

Similarly, we define

$$\langle L(f(x)), g(x) \rangle = \int_D L(f(x)) g(x) dx \quad (5)$$

The first step in calculating the integral, using Method of Moments, is to expand f into a sum of weighted basis functions $f_1, f_2, f_3; \dots$, in the domain of L, as

$$f(x) = \sum_n \alpha_n f_n \quad (6)$$

Testing functions denoted $w_1, w_2, w_3; \dots$. are defined in the range of L. These testing functions are used for all values of n. Using the inner product defined in (5), we obtain

$$\sum_n \langle w_m, L(f_n) \rangle = \langle w_m, g(x) \rangle \quad \text{form} = 1,2,3, \dots \quad (7)$$

Expanding (7) over the values of m and n=1,2,3,…, the following matrix equation is then obtained

$$\begin{bmatrix} \langle w_1, L(f_1) \rangle & \langle w_1, L(f_2) \rangle & \dots & \langle w_1, L(f_n) \rangle \\ \langle w_2, L(f_1) \rangle & \langle w_2, L(f_2) \rangle & \dots & \langle w_2, L(f_n) \rangle \\ \dots & \dots & \dots & \dots \\ \langle w_m, L(f_1) \rangle & \langle w_m, L(f_2) \rangle & \dots & \langle w_m, L(f_n) \rangle \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \dots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \langle w_1, g \rangle \\ \langle w_2, g \rangle \\ \dots \\ \langle w_m, g \rangle \end{bmatrix} \quad (8)$$

In a simpler form,

$$[L_{mn}][\alpha_n] = [G_m], \quad (9)$$

and the solution for the unknown coefficients is then

$$[\alpha_n] = [L_{mn}]^{-1}[G_m] \quad (10)$$

In our calculations, we will choose the test function wm equal to the basis function f_n, which is known as Garlekin method. The determination of matrix $[L_{mn}]$ is straightforward, and its inverse is easy to obtain either analytically or numerically. Once this is done, the α_n coefficients are obtained, and the solution for f is found.

It is good to note here that choosing the appropriate basis/test function is necessary to get fast to the accurate solution.

2.1 Example 1

Considering the following second order differential equation defined by

$$\frac{d^2 f(x)}{dx^2} = x^2 \quad (11)$$

defined over the domain $D = [0, 1]$ with the following boundary conditions $f(0) = f(1) = 0$. Starting by choosing the basis function, let us choose

$$f_n = x - x^{n+1} \quad (12)$$

It is clear from (12) that the chosen basis functions meet the boundary conditions and can be considered as a solution to the problem. Substituting (12) into (6), the left-hand side elements of (7), which are the elements of the matrix $[L_{mn}]$, are found to be

$$\begin{aligned} L_{mn} &= \langle w_m, L(f_n) \rangle = \int_0^1 (x - x^{m+1}) \left(\frac{d^2 (x - x^{n+1})}{dx^2} \right) dx \\ &= \int_0^1 (x - x^{m+1})(-n(n+1)x^{n-1}) dx \\ L_{mn} &= \frac{-m}{n+m+1} \end{aligned} \quad (13)$$

In the same manner, we compute the elements of the matrix $[G_m]$, defined in (7), which are found to be

$$G_m = \langle w_m, g \rangle = \int (x - x^{m+1}) x^2 dx = \frac{m}{4(m+4)} \quad (14)$$

Then, we start by choosing a value for n starting with $n = 1$ until convergence of the solution. For the case when $n = m = 1$, $L_{11} = 1/3$; $G_1 = 1/5, \alpha_1 = 3/5$, and f(x) is given by

$$f(x) = \frac{3}{5}(x - x^2) \quad (15)$$

It is clear from (15), that the function $f(x)$ does not meet the original differential equation defined in (11). Accordingly, we need to increase the value of n.

For $n = m = 2$, the solution for $f(x)$ gives

$$f(x) = \frac{1}{10}(x - x^2) - \frac{1}{3}(x - x^3) \quad (16)$$

Again, (16) does not meet the original equation defined in (11).

For $n = m = 3$, the solution of $f(x)$ is given by

$$f(x) = \frac{1}{12}(x - x^4) \tag{17}$$

The function $f(x)$ given in (17) meets the boundary conditions and (11), and accordingly it is the correct solution of the problem.

2.2 Example 2

Considering the following third order differential equation defined by

$$2\frac{d^3f(x)}{dx^3} + 4\frac{d^2f(x)}{dx^2} - 8f(x) = x^4 - 6x^2 - 14x \tag{18}$$

defined over the domain $D = [0, 2]$ with the following boundary conditions $f(0) = f(2) = 0$. Starting by choosing the basis function, let us choose

$$f_n = 2\left(\frac{x}{2}\right)^n - x \tag{19}$$

It is clear from (19) that the chosen basis functions meet the boundary conditions and can be considered as a solution to the problem. Substituting (19) into (6), the left-hand side elements of (7), which are the elements of the matrix $[L_{mn}]$, are found to be

$$L_{mn} = \langle w_m, L(f_n) \rangle$$

$$L_{mn} = \int_0^2 \left(2\left(\frac{x}{2}\right)^m - x\right)\begin{pmatrix} \frac{n(n-1)(n-2)}{2}\left(\frac{x}{2}\right)^{n-3} + \\ 2n(n-1)\left(\frac{x}{2}\right)^{n-2} - 16\left(\frac{x}{2}\right)^n + 8x \end{pmatrix} dx \tag{20}$$

$$L_{mn} = A + B + C + D - \frac{64}{n+m+1} + \frac{64}{n+2} + \frac{64}{m+2} - \frac{63}{3} \tag{21}$$

In (21),

$$A = \begin{cases} \frac{2n(n-1)(n-2)}{n+m-2} & \text{for } n+m-2 \neq 0 \\ 0 & \text{for } n+m-2 = 0 \end{cases},$$

$$B = \begin{cases} -2n(n-2) & \text{for } n \neq 1 \\ 0 & \text{for } n = 1 \end{cases},$$

$$C = \begin{cases} \frac{8n(n-1)}{n+m-1} & \text{for } n+m-1 \neq 0 \\ 0 & \text{for } n+m-1 = 0 \end{cases},$$

$$D = \begin{cases} -8(n-2) & \text{for } n \neq 0, 1 \\ 0 & \text{for } n = 0, 1 \end{cases} \tag{22}$$

In the same manner, we compute the elements of the matrix $[G_m]$, defined in (7), which we found to be

$$G_m = \langle w_m, g \rangle = \int \left(2\left(\frac{x}{2}\right)^m - x\right)(x^4 - 6x^2 - 14x) \tag{23}$$

$$G_m = \frac{64}{m+5} - \frac{96}{m+3} - \frac{112}{m+2} + \frac{153}{3} \tag{24}$$

Then, we start by choosing a value for n starting with $n = 4$ until convergence of the solution. The solution converges for this value. In fact, for $n = m = 4$, $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$, and $\alpha_4 = -1$. For the case when $n = m = 5$, $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0, \alpha_4 = -1$, and $\alpha_5 = 0$. Accordingly, $f(x)$ is given by

$$f(x) = -2\left(\frac{x}{2}\right)^4 + x = x - \frac{x^4}{8} \tag{25}$$

It is clear from (25) that the function $f(x)$ does meet the original differential equation defined in (18) and the two boundary conditions of the problem.

3 CONCLUSION

This paper presented, in simplified steps, the procedure of using the Method of Moments in solving simple differential equations of different orders. The considered examples show the detailed calculation process and make it easier to understand the solution procedure. As can be seen, this approach could be easily used to solve mathematical problems and equations. According to the type of the equation, the solution of the Moment Method will vary to accommodate for the change in the given problem.

REFERENCES

[1] W. C. Gibson, *The Method of Moments in Electromagnetics*, Chapman & Hall/ CRC, Taylor & Francis group, 2008.

[2] R. F. Harrington, *Field Computation by Moment Methods*, Krieger Publishing Co., Inc., 1968.

[3] A. R. Djordjevic and T. K. Sarkar, "A Theorem on the Moment Methods," *IEEE Transactions on Antennas and Propagation*, vol. 35, no. 3, pp. 353-355, March 1987.

[4] E. H. Newman, "Simple Examples of the Method of Moments in Electromagnetics," *IEEE Transactions on Education*, vol. 31, no. 3, pp. 193-200, August 1998.

[5] N.F.O. Serteller, A.G. Ak, G. Kocyigit, and T.C. Akinci, "Experimental Study of Moment Method for Undergraduates in Electromagnetic", *Journal of Electronics and Electrical Engineering*, vol. 3, no. 3, pp. 115-118, 2011